MULTISCALE HOMOGENIZATION OF MONOTONE OPERATORS

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(Communicated by Giuseppe Buttazzo)

Abstract. In this paper we prove a generalization of the iterated homogenization theorem for monotone operators, proved by Lions et al. in [20] and [21]. Our results enable us to homogenize more realistic models of multiscale structures.

1. Introduction. In connection with the analysis of the macroscopic properties of composites one often considers a class of partial differential equations of the form

\[ -\text{div}(c^*_h(x,Du_h)) = f \quad \text{on } \Omega, \quad u_h \in W^{1,p}_0(\Omega). \]

Here, \( c^*_h \) is increasingly oscillating as \( h \to \infty \), \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \), \( 1 < p < \infty \), \( 1/p + 1/q = 1 \) and \( f \in W^{-1,q}(\Omega) \). An interesting problem is to study the asymptotic behavior of the solutions \( u_h \), as \( h \to \infty \). In many important cases it turns out that \( u_h \) converges weakly in \( W^{1,p}_0(\Omega) \) to the solution \( u \) of a so-called homogenized problem

\[ -\text{div}(c^*(Du)) = f \quad \text{on } \Omega, \quad u \in W^{1,p}_0(\Omega). \]

A first mathematical result concerning this problem was given by De Giorgi and Spagnolo in [17] for the linear case when \( c^*_h \) is of the form \( c^*_h(x,\xi) = a(hx,\xi) = A(h\xi)\xi \), where \( A(\cdot) \) is a periodic and bounded matrix. Later on the treatment of this problem was significantly simplified by Murat and Tartar. Important contributions were also given by Bakhvalov, Bensoussan, Lions and Papanicolaou. Chiado Piat and De Franceschi [12] investigated the case when \( a \) satisfies suitable monotonicity, continuity, coerciveness and growth conditions in the second variable. Associated variational problems have been considered by many authors in the framework of \( \Gamma \)-convergence. For more information we refer to the literature, see e.g., the books [7], [13], [18]. We also would like to refer to a recent paper [16] of Francfort, Murat and Tartar on \( H \)-convergence of monotone operators.

2000 Mathematics Subject Classification. 35B27, 35J60, 73B27.

Key words and phrases. G-convergence, Iterated homogenization, monotone operators.
In the study of multiscale structures the function \( c^*_h \) often takes the form
\[
c^*_h(x,\xi) = a(hx, h^2 x, \ldots, h^m x, \xi),
\]
where \( a \) is periodic in the first \( m \) variables. In this case one usually speaks about \textit{multiscale homogenization}. Problems of this type were first considered by Bruggeman [9] in the 30’s. In 1978 Bensoussan, Lions and Papanicolaou [6] proved a result for linear operators which later has been known as the \textit{iterated homogenization theorem}. Roughly speaking, the result states that the effective properties are found by first homogenizing the medium on the finest microlevel, next on the second level, and so on. This theorem has been an indispensable tool in the construction of structures with extreme effective material properties. Concerning this topic we refer to the collection of classical papers in [11], where the introduction gives a good selection of references (see also [23]). A more general version of the iterated homogenization theorem was obtained by Allaire and Briane [2] in the notion of multiscale convergence (which generalizes the concept of 2-scale convergence introduced by Nguetseng [30] and further developed by Allaire [1] and others). Braides and Lukkassen considered the corresponding \( \Gamma \)-convergence results for convex reiterated problems in [8] and [22]. Later on Müller, Braides and Defranceschi generalized this result to standard non-convex vector valued functions, i.e. systems (see [7], page 215). The lack of existence of a similar \( \Gamma \)-convergence result for non-standard Lagrangians was demonstrated in [24], where it was found that all limits of \( \Gamma \)-converging subsequences are sharply bounded between two Lagrangians whose representation can be found iteratively by a more general scheme than in the standard case. The iterated homogenization theorem for monotone operators, was proved by Lions, Lukkassen, Persson and Wall in [20] and [21] in terms of \( G \)-convergence. These papers also presented results concerning multiscale convergence of monotone operators (the proof of these results contained some gaps which were closed in Lukkassen, Nguetseng and Wall [27]). We refer to [10] for reiterated homogenization of degenerate nonlinear elliptic equations. For more general information on reiterated homogenization and its application, we refer to the survey article [25] (see also [3], [4], [5], [15] and [29] regarding some recent developments in this theory).

In this paper we prove some homogenization results for monotone operators which enable us to homogenize realistic multiscale structures that are much more complicated than those of the form (2) (see Theorem 3.1). For example, we may find the effective properties of composite structures where the only information is that the material properties \( G \)-converge on each microlevel separately (see Corollary 2). Our results also make it easier to prove the original iterated homogenization theorem for monotone operators more directly. In fact, by the methods used in [20] and [21] the proof becomes rather cumbersome when \( m \) is an arbitrary integer. Therefore, for the readers convenience, the theorem was only proved for the case \( m = 2 \). However, by our new results, this theorem follows easily, even for arbitrary values of \( m \) (see Corollary 1).

The paper is organized as follows. We have collected some notation and preliminary results in Section 2. The main results are presented in Section 3. Finally, in Section 4 we give the detailed proofs of our main results.

2. Preliminaries. Let \( Y \) and \( Z \) be open bounded rectangles in \( \mathbb{R}^n \), \( |E| \) denotes the Lebesgue measure of the set \( E \subset \mathbb{R}^n \) and \((\cdot,\cdot)\) is the Euclidean scalar product on

\footnote{By our knowledge, this happened to be the last paper of J. L. Lions.}
\[ \mathbb{R}^n. \text{ By } W^{1,p}_{\text{per}}(Y) \text{ we denote the set of every function } u \in W^{1,p}(Y) \text{ with mean value zero which has the same trace on opposite faces of } Y. \text{ Every function } u \in W^{1,p}_{\text{per}}(Y) \text{ can be extended by periodicity to a function in } W^{1,p}_{\text{loc}}(\mathbb{R}^n) \text{ (in this paper we will not make any distinction between the original function and its extension).} \]

We now state two lemmas whose proofs can be found in e.g. [31] and [7] page 229.

**Lemma 2.1.** Let \( g \in L^q(Y, \mathbb{R}^n) \) be a function such that \( \int_Y (g,Dv) \, dx = 0 \) for every \( v \in H^1_{\text{per}}(Y) \). Then \( g \) can be extended by periodicity to an element of \( L^q_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n) \), still denoted \( g \), such that \( -\text{div}g = 0 \) in \( \mathcal{D}'(\mathbb{R}^n) \).

**Lemma 2.2.** Let \( 1 < p < \infty \). Let \( \psi \) be a sequence in \( L^p(\Omega, \mathbb{R}^n) \) which converges weakly to \( v \), \( (\text{div}v_n) \) converges to \( -\text{div}v \) in \( W^{-1,q}(\Omega) \) and let \( u_n \) be a sequence which converges weakly to \( u \) in \( W^{1,p}(\Omega) \). Then

\[
\int_{\Omega} (v_n,Du_n)\phi \, dx \to \int_{\Omega} (v,Du)\phi \, dx,
\]

for every \( \phi \in C_0^\infty(\Omega) \).

The following Lemma is a generalization of the well-known fact that a periodic function converges weakly to its mean value as the oscillation increases. The proof can be found in [26].

**Lemma 2.3.** Let \( 1 \leq p \leq \infty \) and let \( \psi \) be a function in \( L^p(\Omega, \mathbb{R}^n) \) which is periodic for \( m = 1, 2, \ldots \). Moreover, suppose that \( \psi_n \to \psi \) weakly in \( L^p(Y) \) (weak-* if \( p = \infty \)) as \( n \to \infty \). Let \( \omega_n \) be defined by \( \omega_n(x) = \psi_n(hx) \). Then, as \( h \to \infty \) it holds that

\[
\omega_n \to \frac{1}{|Y|} \int_Y \psi(x) \, dx
\]

weakly in \( L^p(\Omega) \) (weak-* if \( p = \infty \)).

Recall that if \( \Omega \) is an open bounded convex set and \( 1 \leq p < \infty \) then the Poincaré-Wirtinger inequality states that there exists a constant \( K > 0 \) such that

\[
\|u - M_\Omega(u)\|_{L^p(\Omega)} \leq K \|Du\|_{L^p(\Omega, \mathbb{R}^n)},
\]

for every \( u \in W^{1,p}(\Omega) \), where \( M_\Omega(u) \) denotes the mean value of \( u \) over \( \Omega \). A corollary of this is that \( \|D\|_{L^p(\Omega, \mathbb{R}^n)} \) defines an equivalent norm on \( W^{1,p}_{\text{per}}(Y) \), i.e. there exists finite constants \( k_1, k_2 > 0 \) such that

\[
k_1 \|D\|_{L^p(\Omega, \mathbb{R}^n)} \leq \|u\|_{W^{1,p}_{\text{per}}(Y)} \leq k_2 \|D\|_{L^p(\Omega, \mathbb{R}^n)}. \tag{3}
\]

3. **Main results.** Let \( C_1, C_2, p, \alpha \) and \( \beta \), be real constants such that \( 0 < C_1, C_2 < \infty \), \( 1 < p < \infty \), \( 0 < \alpha \leq \min \{1,p-1\} \) and max \{\( p, 2 \)\} \( \leq \beta < \infty \). Consider the class \( \mathcal{B} = \mathcal{B}(C_1, C_2, p, \alpha, \beta) \) consisting of all functions \( a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) satisfying the following measurability, boundedness, continuity and monotonicity assumptions:

\[ a(\cdot, \xi) \text{ is Lebesgue measurable for every } \xi \in \mathbb{R}^n, \tag{4} \]

\[ a(x,0) = 0, \tag{5} \]

\[ |a(x,\xi_1) - a(x,\xi_2)| \leq C_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha, \tag{6} \]

\[ (a(x,\xi_1) - a(x,\xi_2), \xi_1 - \xi_2) \geq C_2(1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta, \tag{7} \]
for every \( x, \xi_1, \xi_2 \in \mathbb{R}^n \).

By (5), (6) and (7) it follows that
\[
|a(x, \xi)| \leq C \left( 1 + |\xi|^{p-1} \right),
\]

(8)
\[
|\xi|^p \leq C' \left( 1 + (a(x, \xi)), \xi \right),
\]

(9)

hold for all \( x \in \mathbb{R}^n \) and every \( \xi \in \mathbb{R}^n \) for some constant \( C \) and \( C' \). The proof of (8) follows by (5), (6) and the fact that \( |\xi|^p \leq (1 + |\xi|)^\alpha \), which gives
\[
|a(x, \xi)| \leq C_1 (1 + |\xi|)^{p-1} \leq C_1 \left( (2 |\xi|)^{p-1} + (2)^{p-1} \right) \leq C_1 2^{p-1} \left( |\xi|^{p-1} + 1 \right).
\]

The proof of (9) can be found in [7, p. 231].

Remark 1.

The proof of (9) can be found in [7, p. 231].

Let \( \{a_h\}_{h=1}^\infty \) be a sequence in \( \mathfrak{B} \) and consider the problem:
\[
\begin{cases}
\int_\Omega (a_h(x, Du_h) , D\phi) = (f, \phi), \text{ for every } \phi \in W^{1,p}_0(\Omega) \\
u_h \in W^{1,p}_0(\Omega),
\end{cases}
\]

(10)

where \( f \in W^{-1,q}(\Omega) \) (\( q \) is the exponential dual of \( p \), i.e. \( 1/p + 1/q = 1 \)) and \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \). By the Hartman-Stampacchia Theorem (see [19]) there exists a unique solution \( u_h \) for each \( h \in \mathbb{N} \). We say that \( a_h \) \( G \)-converges in \( \Omega \) to \( a \in \mathfrak{B} \), denoted \( a_h \rightharpoonup^G a \), if
\[
u_h \rightharpoonup u \text{ weakly in } W^{1,p}_0(\Omega),
\]
\[
a_h(x, Du_h) \rightharpoonup a(x, Du) \text{ weakly in } L^q(\Omega, \mathbb{R}^n),
\]

for every \( f \in W^{-1,q}(\Omega) \), where \( u \) is the solution of the problem
\[
\begin{cases}
\int_\Omega (a(x, Du), D\phi) = (f, \phi), \text{ for every } \phi \in W^{1,p}_0(\Omega) \\
u \in W^{1,p}_0(\Omega).
\end{cases}
\]

(11)

Remark 1. In many practical examples it turns out that possible \( G \)-limits of a sequence \( a_h \in \mathfrak{B}(C_1, C_2, p, \alpha, \beta) \) belong to some different class \( \mathfrak{B}(C_1, C_2, p, \gamma, \beta) \) where \( 0 < \gamma \leq \alpha \). However, this is not an obstacle with respect to the definition of \( G \)-convergence since we easily see that \( a_h \) belongs to the same class. Indeed, the fact that \( 1 + [\xi_1] + [\xi_2] > [\xi_1 - \xi_2] \) gives that
\[
|a_a(x, \xi_1) - a_h(x, \xi_2)| \leq C_1 (1 + [\xi_1] + [\xi_2])^{p-1} [\xi_1 - \xi_2]^{\alpha} = C_1 (1 + [\xi_1] + [\xi_2])^{p-1} \gamma^{-\alpha} [\xi_1 - \xi_2]^{\gamma-\alpha} \leq C_1 (1 + [\xi_1] + [\xi_2])^{p-1} \gamma^{-\alpha} [\xi_1 - \xi_2]^{\gamma-\alpha} \leq C_1 (1 + [\xi_1] + [\xi_2])^{p-1} \gamma^{-\alpha} [\xi_1 - \xi_2]^{\gamma-\alpha} \leq C_1 (1 + [\xi_1] + [\xi_2])^{p-1} \gamma^{-\alpha} [\xi_1 - \xi_2]^{\gamma-\alpha}.
\]

Hence, \( a_h \in \mathfrak{B}(C_1, C_2, p, \gamma, \beta) \). The easiest example is when \( a_h(x, \xi) = c(hx, \xi) \), where \( c \in \mathfrak{B}(C_1, C_2, p, \alpha, \beta) \) and \( c(\cdot, \xi) \) is \( Y \)-periodic for some cell \( Y \). In this case \( \gamma = \alpha/(\beta - \alpha) \).

Let \( \Omega_i \subset \mathbb{R}^n \), \( i = 1, \ldots, N \) be a collection of disjoint open sets such that \( \mathbb{R}^n \setminus \bigcup_{i=1}^N \Omega_i = 0 \) and \( \partial \Omega_i = 0 \) and let \( \mathcal{D} \) be the collection of each function \( a : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) which fulfills conditions of type (4), (5), (6) and (7) in the second and third variable in addition to some piecewise continuity condition in the first variable. More precisely, we assume that

1. \( a(y, \cdot, \xi) \) is \( Z \)-periodic and Lebesgue measurable for every \( \xi \in \mathbb{R}^n \) and every \( y \in \mathbb{R}^n \)
Theorem 3.1. Let \( c, c_h \in \mathcal{D} \) and assume that for each \( y \in \mathbb{R}^n \) we have that \( c_h^y \overset{G}{\rightharpoonup} c^y \). Then, for the sequence \( c_h^y \) given by \( c_h^y(x, \xi) = c_h(x, hx, \xi) \) it holds that

\[
c^y_h(x, \xi) \rightharpoonup c^y(x, \xi) \quad \text{def} \quad \frac{1}{|Z|} \int_Z c(x, z, \xi + Dv(z)) \, dz,
\]

where \( v \) is the unique solution of the cell-problem

\[
\begin{align*}
\int_Z (c(x, z, \xi + Dv(z)), D\phi) \, dz &= 0 \quad \text{for every } \phi \in W^{1,p}_0(Z), \\
v &\in W^{1,p}_0(Z).
\end{align*}
\]

Moreover, \( c^* \in \mathcal{B}(C_1, C_2, p, \gamma, \beta) \) for \( \gamma = \alpha/(\beta - \alpha) \) and \( c^* \) is of the form

\[
c^*(y, \xi) = \sum_{i=1}^{N} \chi_{\Omega_i}(y) c^*_i(y, \xi),
\]

where \( c^*_i(\cdot, \xi) \) satisfies the continuity condition

\[
|c^*_i(y_1, \xi) - c^*_i(y_2, \xi)|^q \leq \omega(|y_1 - y_2|)(1 + |\xi|^p).
\]

More generally, let

\[
a : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

and assume that \( a(x^1, \ldots, x^{m+1}, \xi) \) is \( Y_k \)-periodic in the \( k \)-th variable for \( k = 2, \ldots, m+1 \), Lebesgue measurable in the \( (m+1) \)-th variable, \( a(x^1, \ldots, x^{m+1}, 0) = 0 \) and

\[
|a(x^1, \ldots, x^{m+1}, \xi_1) - a(x^1, \ldots, x^{m+1}, \xi_2)| \leq C_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha,
\]

(16)

\[
(a(x^1, \ldots, x^{m+1}, \xi_1) - a(x^1, \ldots, x^{m+1}, \xi_2), \xi_1 - \xi_2) \geq C_2(1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta.
\]

(17)
Moreover, we assume that \( a \) satisfies the following piecewise continuity condition in the \( j \)-th variable for \( j = 1, \ldots, m \):

\[
a(x^1, \ldots, x^{m+1}, \xi) = \sum_{i=1}^{N} \chi_{\Omega_i}(y) a_i^j(x^1, \ldots, x^{m+1}, \xi),
\]

where \( a_i^j \) satisfies the continuity condition

\[
\left| a_i^j(x^1, \ldots, x_1^j, \ldots, x^{m+1}, \xi) - a_i^j(x^1, \ldots, x_1^j, \ldots, x^{m+1}, \xi) \right| \leq \omega \left| x_i^j - x_i^j \right| (1 + |\xi|^p).
\]

(18)

By using Theorem 3.1 we easily obtain the following result.

**Corollary 1.** For the sequence \( a_h \) given by \( a_h(x, \xi) = a(x, hx, \ldots, h^m x, \xi) \) it holds that \( a_h \overset{G}{\to} \alpha[a]\), where \( a[0] = a \) and \( a[m-k] \) is found iteratively according to the following scheme:

\[
a[m-k](x_1, \ldots, x_{k+1}, \xi) = \frac{1}{|Y_{k+2}|} \int_{Y_{k+2}} a[m-k-1](x_1, \ldots, x_{k+1}, y, \xi + Du(y)) dy,
\]

where \( v \) is the unique solution of the cell-problem

\[
\begin{cases}
\int_{Y_{k+2}} (a[m-k-1](x_1, \ldots, x_{k+1}, y, \xi + Du(y)), D\phi) dy = 0 \text{ for every } \phi \in W^{1,p}_{per}(Y_{k+2}), \\
v \in W^{1,p}_{per}(Y_{k+2}).
\end{cases}
\]

**Remark 2.** It is certainly important that the function

\[
x \mapsto a_h(x, Du(x)) = a(x, hx, \ldots, h^m x, Du(x))
\]

and the other functions above are measurable. Fortunately, this is ensured by the fact that the function \( f(x^{m+1}, y) = a(x^1, \ldots, x^{m+1}, \xi) \), where \( y = (x^1, \ldots, x^m, \xi) \), is a piecewise Carathéodory function, i.e. \( f \) is piecewise continuous in the \( y \)-variable and measurable in \( x^{m+1} \)-variable.

**Example 1.** As an example, let \( \Omega \subset \mathbb{R}^2 \), and let \( a(x, \xi) = \lambda(x)I \), where \( \lambda(x) = \chi(x)\lambda_o + (1-\chi(x))\lambda_I \) is the local conductivity of a two-component square honeycomb structure. Here, \( \chi \) denotes the characteristic function of the exterior part of the honeycomb, and \( \lambda_I \) and \( \lambda_o \) are the conductivities of the interior and exterior part, respectively. Due to symmetry it is possible to show that \( a(hx, \xi) \overset{G}{\to} \lambda[1]I \), for some positive constant number \( \lambda[1] \). For the interesting case \( \lambda_I \approx 0 \), the following approximation formulae has been proposed by Meidell [28]:

\[
\lambda[1] \approx \frac{50\lambda_o (1-v_I)}{(1+v_I) (-9v_I^2 + 9v_I + 50)},
\]

(19)

where \( v_I \) denotes the volume fraction of the material with conductivity \( \lambda_I \). This formula is valid with error less than 0.8%. Consider now a new square honeycomb structure where the volume fraction of the interior part is denoted \( v'_I \) and the exterior part is replaced by the previously mentioned square honeycomb structure with a much smaller length-scale (see Figure 1) for which the local conductivity \( \lambda_h(x) \) takes the form \( \lambda_h(x) = \gamma(hx, h^2 x) \) where \( \gamma(x, y) = \chi'(x)\lambda(y) + (1-\chi'(x))\lambda_I \). By Corollary 1 we obtain that \( a_h(x, \xi) = \lambda(hx, h^2 x)I \overset{G}{\to} \lambda[2]I \), for some positive constant number \( \lambda[2] \). In particular, for the case \( \lambda_I \approx 0 \), we obtain that

\[
\lambda[2] \approx \frac{50\lambda[1] (1-v'_I)}{(1+v'_I) (-9(v'_I)^2 + 9v'_I + 50)} \approx
\]
\[
\frac{2500 \lambda_0 (1 - v_f)(1 - v'_f)}{(1 + v_f)(1 + v'_f) (-9v_f^2 + 9v_f + 50) (-9v'_f^2 + 9v'_f + 50)}.
\]

Moreover, according to the above mentioned accuracy, formula (20) is valid with error less than 1.61%.

Figure 1. Example of a multi-scale material structure with two microlevels.

Proof of Corollary 1. First we observe by repeated use of (15) in Theorem 3.1 that \(a[y]\) inherit the piecewise continuity condition of \(a\) given by (18). Moreover, let \(x^1, \ldots, x^{m-2}\) be fixed and let \(c'_h(x, \xi) = c_h(y, x, \xi) = a(x^1, \ldots, x^{m-2}, y, x, hx, \xi)\). A simple application of Theorem 3.1 gives that \(c'_h \overset{G}{\to} c^g\), where

\[
c^g(x, \xi) = c(y, x, \xi) = a[1](x^1, \ldots, x^{m-2}, y, x, \xi).
\]

Moreover, using Theorem 3.1 once more, we obtain for the function \(c^*(y, x, \xi) = a[1](x^1, \ldots, x^{m-2}, x, x, hx, h^2x, \xi)\) that \(c^* \overset{G}{\to} c^*, \) where \(c^*(y, x, \xi) = a[2](x^1, \ldots, x^{m-2}, x, x, hx, h^2x, \xi)\). For the next iteration we fix \(x^1, \ldots, x^{m-3}\) and let

\[
c'_h(x, \xi) = c_h(y, x, \xi) = a(x^1, \ldots, x^{m-3}, y, x, hx, h^2x, \xi)
\]

and use Theorem 3.1 to obtain that \(c'_h \overset{G}{\to} c^*, \) where

\[
c^*(x, \xi) = a[1](x^1, \ldots, x^{m-3}, x, x, hx, h^2x, h^3x, \xi)
\]

and \(c^*(x, \xi) = a[3](x^1, \ldots, x^{m-3}, x, x, hx, h^2x, h^3x, \xi)\). Continuing this procedure \(m - 3\) more times, the stated result follows.

Proposition 1. Let \(\xi \in \mathbb{R}^n, c'_h, c^* \in \mathcal{B}\), and let \(v_h, v \in W^{1,p}_{\text{per}}(Z)\), with zero average value, be the solutions of the problems

\[
\int_Z (c'_h(z, \xi + Dv_h(z)), D\phi) \, dz = 0 \text{ for every } \phi \in W^{1,p}_{\text{per}}(Z),
\]

\[
\int_Z (c^*(z, \xi + Dv(z)), D\phi) \, dz = 0 \text{ for every } \phi \in W^{1,p}_{\text{per}}(Z),
\]

respectively. If \(c'_h \overset{G}{\to} c^*\) in \(Z\), then it holds that

\[
v_h \rightharpoonup v \text{ weakly in } W^{1,p}_{\text{per}}(Z),
\]

\[
c'_h(\cdot, \xi + Dv_h(\cdot)) \rightharpoonup c^*(\cdot, \xi + Dv(\cdot)) \text{ weakly in } L^p(Z, \mathbb{R}^n).
\]
Proposition 2. Let $\Omega_1$ and $\Omega_2$ be disjoint open bounded subsets of $\Omega$ such that $|\Omega\setminus(\Omega_1 \cup \Omega_2)| = 0$ and $|\partial \Omega_1| = 0$. If $c^*_h \lesssim c^*$ in $\Omega_1$ and $\Omega_2$, separately, then $c^*_h \lesssim c^*$ in $\Omega$.

Let $b_h, c_h, b, c \in \mathcal{D}$ and $Y$-periodic in the first variable (and the second variable), and assume that for each $y \in \mathbb{R}^n$ we have that $b^h_y \lesssim b^y$ and $c^*_h \lesssim c^y$. Let $A$ be an open $Y$-periodic subset of $\mathbb{R}^n$. Moreover, let

$$a_h(x, \xi) = \tilde{a}_h(x, h x, \ldots, h^m x, \xi),$$

where $\tilde{a}_h = \tilde{a}_h^{[m]}$ is found iteratively according to the following scheme:

$$\tilde{a}_h^{[0]}(y, z, \xi) = b(y, z, \xi),$$

$$\tilde{a}_h^{[k]}(x^{k+2}, \ldots, x^1, \xi) = \chi_A(x^{k+2}) \tilde{a}_h^{[k-1]}(x^{k+1}, \ldots, x^1, \xi) + (1 - \chi_A(x^{k+2})) c_h(x^{k+1}, x^k, \xi).$$

Similarly, let $a'_h(x, \xi) = a(x, h x, \ldots, h^m x, \xi)$, where $a = \tilde{a}^{[m]}$ is found iteratively according to the following scheme:

$$\tilde{a}_h^{[0]}(y, z, \xi) = b(y, z, \xi),$$

$$\tilde{a}_h^{[k]}(x^{k+2}, \ldots, x^1, \xi) = \chi_A(x^{k+2}) \tilde{a}_h^{[k-1]}(x^{k+1}, \ldots, x^1, \xi) + (1 - \chi_A(x^{k+2})) c(x^{k+1}, x^k, \xi).$$

By Corollary 1 we have that $a'_h \lesssim a^{[m]}$ where $a^{[m]}$ is defined in the same Corollary. By taking Proposition 2 into account and arguing as we did in the proof of Corollary 1 we are able to obtain the following more general result.

Corollary 2. For the sequence $a_h$ given by (23) it holds that $\tilde{a}_h \lesssim a^{[m]}$.

4. Proofs. In this section we have collected the remaining proofs of the main results.

Proof of Proposition 1. By (9), (21) (with $\phi = v_h$), (5), (6) and Hölder’s inequality we obtain that

$$\int_Z |\xi + Dv_h|^p \, dx \leq C' \int_Z 1 + (c_h^*(x, \xi + Dv_h), \xi + Dv_h) \, dx =$$

$$C' \int_Z 1 + (c_h^*(x, \xi + Dv_h), \xi) \, dx \leq$$

$$C' \int_Z 1 + |c_h^*(x, \xi + Dv_h, |\xi|) \, dx \leq C' \left(|Z| + C |\xi| \left(|Z| + \int_Z |\xi + Dv_h|^{p-1} \, dx\right)\right) \leq$$

$$C' \left(|Z| + C |\xi| \left(|Z| + |\xi| \int_Z |\xi + Dv_h|^p \, dx \right)^{\frac{1}{p}}\right) \leq$$

$$C'' \left(1 + \left(\int_Z |\xi + Dv_h|^p \, dx\right)^{\frac{1}{p}}\right).$$
Let us now define

\[ v_h = c_0^h(\xi + Dv_h). \]

By (5), (6) and (25) we have that \( v_h \) is bounded in \( L^q(Z, \mathbb{R}^n) \). Indeed,

\[
\begin{align*}
\int_Z |v_h|^q \, dx &= \int_Z |c_0^h(x, \xi + Dv_h)|^q \, dx \\
&\leq C_1 \int_Z (1 + |\xi + Dv_h|)^{q(\rho - 1)} |\xi + Dv_h|^{q\rho_0} \, dx \\
&\leq C_1 \int_Z (1 + |\xi + Dv_h|)^{q(\rho - 1)} \, dx = C_1 \int_Z (1 + |\xi + Dv_h|)^p \, dx \\
&\leq C_1 2^p \int_Z (1 + |\xi + Dv_h|)^p \, dx \\
&\leq C,
\end{align*}
\]

where \( C \) is a finite constant independent of \( h \). This means that there exists a subsequence, still denoted by \( (v_h) \), and a vector function \( \eta_* \in L^q(Z, \mathbb{R}^n) \) such that

\[ v_h \rightharpoonup \eta_* \text{ weakly in } W^{1,p}_{\text{per}}(Z). \]  

(26)

Hence, by (21) we find that

\[ \int_Z (\eta_*, D\phi) \, dx = 0 \text{ for every } \phi \in C_0^\infty(Z). \]

If we now could show that

\[ \eta_* = c^*(\xi + D\eta_*) \text{ a.e. in } Z, \]  

(27)

then it follows by the uniqueness of the problem (22) that \( \eta_* = \eta \). In order to prove (28) let \( \tau \in \mathbb{R}^n \), let \( Z' \) be an open bounded set which is strictly contained in \( Z \) and let \( u \in C_0^\infty(Z) \) be such that \( Du = \tau \) in \( Z' \). Associated with the function \( u \) we define a functional \( f_u \in W^{-1,q}(Z) \) by

\[ \langle f_u, \phi \rangle = \int_Z (c^*(z, Du), D\phi) \, dz, \phi \in W^{1,p}_0(Z). \]

(28)

Let \( u_h \in W^{1,p}_0(Z) \) be the solution of the problem

\[ \int_Z (c_0^h(z, Du_h(z)), D\phi) \, dz = \langle f_u, \phi \rangle \text{ for every } \phi \in W^{1,p}_0(Z). \]

(29)
Due to the fact that $c^*_h \overset{G}{\rightarrow} c^*$ in $Z$ we have that

$$u_h \rightharpoonup u \text{ weakly in } W^{1,p}_0(Z),$$

$$c^*_h(x, Du_h) \rightarrow c^*(x, Du) \text{ weakly in } L^q(Z, \mathbb{R}^n).$$

By the monotonicity of $c^*_h$ we have that

$$
\int_Z c^*_h(x, \xi + Dv_h) - c^*_h(x, Du_h) - \xi^+ Du_h \phi \, dx \geq 0
$$

for every $\phi \in C^\infty_0(Z'), \phi \geq 0$. Choosing a subsequence such that both the convergences (26) and (27) occur, we obtain by Lemma 2.2 that

$$
\int_Z (\eta_\ast - c^*(x, \tau), \xi + Du_\ast - \tau) \phi \, dx \geq 0.
$$

Hence, taking into account that the set $Z' \subset Z$ was arbitrarily chosen, we find that

$$(\eta_\ast(x) - c^*(x, \tau), \xi + Du_\ast(x) - \tau) \geq 0$$

for all $x \in Z \setminus A_\tau$ for some set $A_\tau$ with zero Lebesgue measure. In particular, this holds when $\tau$ belongs to a countable dense subset $\{\tau_k\}$ of $Z$. Hence,

$$(\eta_\ast(x) - c^*(x, \tau_k), \xi + Du_\ast(x) - \tau_k) \geq 0$$

for all $x \in Z \setminus A$, where $A = \bigcup_{k=0}^\infty A_{\tau_k}$ has zero Lebesgue measure. Hence, since $c^*(x, \cdot)$ is continuous, this shows that

$$(\eta_\ast(x) - c^*(x, \tau), \xi + Du_\ast(x) - \tau) \geq 0$$

for all $x \in Z \setminus A$ and all $\tau \in \mathbb{R}^n$. Moreover, since $c^*(x, \cdot)$ is monotone and continuous, we have that $c^*(x, \cdot)$ is maximal monotone. Therefore, (31) implies the crucial relation (28). We have now proved the Proposition up to a subsequence of $(v_h)$. By the uniqueness of the solution to the (22) it follows that it is true for the whole sequence, and the proof is complete.

\section*{Proof of Theorem 3.1.}

For the proof of (15) and the fact that $c^* \in \mathcal{B}(C_1, C_2, p, \gamma, \beta)$ we refer to [26].

Let $f \in W^{-1,q}(\Omega)$ and consider the problem

$$
\begin{cases}
\int_\Omega c^*_h(x, Du_h(x)) \, D\phi(x) \, dx = \langle f, \phi \rangle & \text{for every } \phi \in W^{1,p}_0(\Omega), \\
u_h \in W^{1,p}_0(\Omega),
\end{cases}
$$

(32)
along with the limit problem

$$
\begin{cases}
\int_\Omega c^*(x, Du(x)) \, D\phi(x) \, dx = \langle f, \phi \rangle & \text{for every } \phi \in W^{1,p}_0(\Omega), \\
u \in W^{1,p}_0(\Omega).
\end{cases}
$$

(33)

By the definition of $G$-convergence, we only have to prove that

$$u_h \rightharpoonup u \text{ weakly in } W^{1,p}_0(\Omega),$$

$$c^*_h(x, Du_h) \rightarrow c^*(x, Du) \text{ weakly in } L^q(\Omega, \mathbb{R}^n).$$

We divide the proof into several steps.

**Step 1.** Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$, let $\{\Omega^i_k \subset \Omega : i \in I_k\}$ denote a family of disjoint open sets with diameter less than $1/k$ such that $|\Omega \setminus \bigcup_{i \in I_k} \Omega^i_k| = 0$.
and $|\partial \Omega_i^k| = 0$, and such that each $\Omega_i^k$ belongs to some set $\Omega_j$ (defined in Section 3). Let $c_{m}^{k}$ and $c^{k}$ be the function defined by

$$
c_{m}^{k}(y, z, \xi) = \sum_{i \in I_k} \chi_{\Omega_i^k}(y)c_m(y_i^k, z, \xi),
$$

$$
c^{k}(y, z, \xi) = \sum_{i \in I_k} \chi_{\Omega_i^k}(y)c(y_i^k, z, \xi),
$$

where $y_i^k \in \Omega_i^k$ and $\chi_A$ is the characteristic function (indicator function) of the set $A$. In addition, let $c_{h}^{k,m}(x, \xi)$ be given by $c_{h}^{k,m}(x, \xi) = c_{m}^{h}(x, hx, \xi)$ and let $c^{k,m}(y, \xi)$, $c^{k}(y, \xi)$ be given by

$$
ce^{k,m}(y, \xi) = \frac{1}{|Z|} \int_{Z} c_{m}^{k}(y, z, \xi + Du^{k,m}(z)) \, dz,
$$

$$
ce^{k}(y, \xi) = \frac{1}{|Z|} \int_{Z} c^{k}(y, z, \xi + Du^{k}(z)) \, dz,
$$

where $v^{k,m}$ and $v^{k}$ are the unique solutions of the cell-problems

$$
\begin{cases}
\int_{Z}(c_{m}^{k}(y, z, \xi + Du^{k,m}(z)), D\phi) \, dz = 0 & \text{for every } \phi \in W_{per}^{1,p}(Z), \\
v^{k,m} \in W_{per}^{1,p}(Z),
\end{cases}
$$

and

$$
\begin{cases}
\int_{Z}(c^{k}(y, z, \xi + Du^{k}(z)), D\phi) \, dz = 0 & \text{for every } \phi \in W_{per}^{1,p}(Z), \\
v^{k} \in W_{per}^{1,p}(Z),
\end{cases}
$$

respectively. Note in particular that

$$
ce^{k}(y, \xi) = \sum_{i \in I_k} \chi_{\Omega_i^k}(y)c^{*}(y_i^k, \xi).$$

By combining suitable modifications of well-known homogenization techniques with the use of Proposition 1 and Lemma 2.3 we will first prove that

$$c_{h}^{k,h}(x, \xi) \xrightarrow{G} c^{k}(x, \xi).$$

Consider the problem

$$
\begin{cases}
\int_{\Omega}(c_{h}^{k,m}(x, Du^{k,m}(x)), D\phi(x)) \, dx = (f, \phi) & \text{for every } \phi \in W_{0}^{1,p}(\Omega), \\
u_{h}^{k,m} \in W_{0}^{1,p}(\Omega),
\end{cases}
$$

along with the limit problems

$$
\begin{cases}
\int_{\Omega}(c^{k,m}(x, Du^{k,m}(x)), D\phi(x)) \, dx = (f, \phi) & \text{for every } \phi \in W_{0}^{1,p}(\Omega), \\
u^{k,m} \in W_{0}^{1,p}(\Omega),
\end{cases}
$$

$$
\begin{cases}
\int_{\Omega}(c^{k}(x, Du^{k}(x)), D\phi(x)) \, dx = (f, \phi) & \text{for every } \phi \in W_{0}^{1,p}(\Omega), \\
u^{k} \in W_{0}^{1,p}(\Omega).
\end{cases}
$$

By (9), (39) (with $\phi = u_{h}^{k,m}$) and Friedrichs inequality we find that

$$
\int_{\Omega} |Du_{h}^{k,m}|^{p} \, dx \leq C' \int_{\Omega} 1 + (c_{h}^{k,m}(x, Du_{h}^{k,m}(x))), Du_{h}^{k,m}) \, dx =
$$
\[ C' \int_{\Omega} 1 \, dx + C \left( f, u_h^{k,m} \right) \leq C''(1 + \left( \int_{\Omega} |Du_h^{k,m}|^p \, dx \right)^{\frac{1}{p}}) \]

for some finite constant \( C'' \), which is independent of \( h \) and \( m \). Hence, we obtain that

\[ \left( \int_{\Omega} |Du_h^{k,m}|^p \, dx \right)^{\frac{1}{p}} \leq C'' \left( \left( \int_{\Omega} |Du_h^{k,m}|^p \, dx \right)^{\frac{1}{p}} + 1 \right), \quad (42) \]

which shows that \( Du_h^{k,h} \) is bounded in \( L^p(\Omega, \mathbb{R}^n) \) (in the exact same way we also find that \( Du_h \) and \( Du^k \) are bounded in \( L^p(\Omega, \mathbb{R}^n) \), which we will use in the next steps). Similarly as in the proof of Proposition 1 we obtain that there exist subsequences, \( (u_h^{k,h}) \), \( (c_h^{k,h}(\cdot, Du_h^{k,h}(\cdot))) \) and functions \( u_* \in W^{1,p}_0(\Omega) \), \( \eta_* \in L^q(\Omega, \mathbb{R}^n) \) such that

\[ u_h^{k,h} \rightarrow u_* \text{ weakly in } W^{1,p}_0(\Omega), \quad (43) \]

\[ c_h^{k,h}(\cdot, Du_h^{k,h}(\cdot)) \rightarrow \eta_* \text{ weakly in } L^q(\Omega, \mathbb{R}^n), \quad (44) \]

as \( h \rightarrow \infty \). Hence, according to (39),

\[ \int_{\Omega_h^i} (\eta_*, D\phi) \, dx = (f, \phi) \text{ for every } \phi \in C_{0}^{\infty}(\Omega_h^i), \quad i \in I_h. \]

If we now could show that

\[ \eta_* = c^k(\cdot, Du_*(\cdot)) \text{ for a.e. } x \in \Omega_h^i, \quad (45) \]

then it follows by the uniqueness of the limit problem (41) that \( u_* = u^k \). To this end we define the function

\[ w_h^{k,m}(x) = (\xi, x) + \frac{1}{h} u^{k,m}(hx), \]

where \( u^{k,m} \) is the solution of (35) for \( y = y_h^i \) (or equivalently for any \( y \in \Omega_h^i \)). By Lemma 2.3 we obtain that

\[ w_h^{k,h} \rightarrow (\xi, \cdot) \text{ weakly in } W^{1,p}(\Omega_h^i), \]

\[ Dw_h^{k,h} \rightarrow \xi \text{ weakly in } L^p(\Omega_h^i, \mathbb{R}^n). \]

Set \( \psi_m = c_m(x_h^i, \cdot, \xi + Du^{k,m}(\cdot)) \) and \( \psi = c(x_h^i, \cdot, \xi + Du^k(\cdot)) \), and note that

\[ \psi_m(hx) = c_h^{k,m}(x, Du_h^{k,m}(x)). \]

Moreover, let \( \omega_h(x) = \psi_h(hx) \). By Proposition 1 \( v^{k,m} \rightarrow v^k \) weakly in \( W^{1,p}(Z) \) and

\[ c_m(x_h^i, \cdot, \xi + Du^{k,m}(\cdot)) \rightarrow c(x_h^i, \cdot, \xi + Du^{k}(\cdot)) \text{ weakly in } L^q(Z, \mathbb{R}^n), \]

i.e. \( \psi_m \rightarrow \psi \) weakly in \( L^q(Z, \mathbb{R}^n) \). According to Lemma 2.3 and (34)

\[ \omega_h = c_h^{k,h}(\cdot, Du_h^{k,h}(\cdot)) \rightarrow c^k(\cdot, \xi) \text{ weakly in } L^q(\Omega_h^i, \mathbb{R}^n). \]

Due to the monotonicity of \( c_h^{k,m}(x, \cdot) \) we have that

\[ \int_{\Omega_h^i} (c_h^{k,h}(x, Du_h^{k,h}(x)) - c_h^{k,h}(x, Du_h^{k,h}(x)), Du_h^{k,h}(x) - Du_h^{k,h}(x)) \phi \, dx \geq 0, \]

for every \( \phi \in C_{0}^{\infty}(\Omega_h^i), \phi \geq 0 \). By Lemma 2.2 we obtain for the limit that

\[ \int_{\Omega_h^i} (\eta_*(x) - c^k(x, \xi)), Du_*(x) - \xi \phi \, dx \geq 0. \]
Hence, similarly as in the proof of Proposition \(1\), we obtain that
\[
(\eta(x) - c^k(x, \xi)) \cdot D\eta(x) - \xi \geq 0 \text{ for a.e. } x \in \Omega_1^\varepsilon \text{ and for every } \xi \in \mathbb{R}^n.
\]
In the special case when \(c_{\eta_1}(y, z, \xi) = c(y, z, \xi)\) we certainly have that \(c^k = c^s\).
Since generally \(c^s \in \mathcal{B}(C_1, C_2, p, \gamma, \beta)\) and satisfies (15), we know that \(c^k\) possesses these properties. In particular, \(c^k\) is monotone and continuous, hence maximal monotone, and the crucial relation (45) follows. We have only proved this for an arbitrary weakly convergent subsequences \((u_{k,h}^n)\) and \(c_{k,h}(x, Du_{k,h}^n(x))\). However, by the uniqueness of the solution of the limit problem (40) it follows that it is true for the whole sequence.

**Step 2.** Let us now prove that \(u_k \to u\) weakly in \(W_0^{1,p}(\Omega)\) (where \(u_k\) and \(u\) are the solutions of (32) and (33)). If \(g \in W_0^{-1,p}(\Omega)\), then
\[
\lim_{h \to \infty} \lim_{k \to \infty} \|g, u_h - u_k\| \leq \lim_{k \to \infty} \lim_{h \to \infty} \|g, u_h - u_k\| + \lim_{k \to \infty} \lim_{h \to \infty} \|g, u^h_k - u^h_k\| + \lim_{k \to \infty} \lim_{h \to \infty} \|g, u^h_k - u^h_k\|,
\]
provided that the limits on the right-hand side exist. In order to prove that \(u_h \to u\) weakly in \(W_0^{1,p}(\Omega)\) we need to show that these limits are zero.

**Term 1.** Let us prove that
\[
\lim_{k \to \infty} \lim_{h \to \infty} \|u_h - u^h_k\| = 0. \quad (46)
\]
Putting \(\phi = u^h_k - u_h\) into (32) and (39) we obtain that
\[
\int_{\Omega} (c_{k,h}^n(x, Du_{h,k}^n(x)) - c_{k,h}^n(x, Du_h(x)), Du_{h,k}^n(x) - Du_h(x)) dx = \\
\int_{\Omega} (c_{h}^n(x, Du_h(x)) - c_{k,h}^n(x, Du_h(x)), Du_{h,k}^n(x) - Du_h(x)) dx.
\]
Now, assume that \(p < \beta\). Due to the fact that \((Du_h)\) and \((Du_{h,k}^n)\) are bounded in \(L^p(\Omega, \mathbb{R}^n)\) we find that
\[
C' \leq \left( \int_{\Omega} \left( 1 + |Du_{h,k}^n| + |Du_h| \right)^p dx \right)^{\frac{p-\beta}{p}}, \quad \left( \int_{\Omega} \left( 1 + |Du_{h,k}^n| + |Du_h| \right)^p dx \right)^{\frac{1}{p}} \leq C''
\]
for some strictly positive finite constants \(C'\) and \(C''\). By using (13) and Hölder’s reversed inequality on the left-hand side and Hölder’s inequality and (14) on the right hand side, we obtain that
\[
C_2 \left( \int_{\Omega} |Du_{h,k}^n - Du_h|^\beta dx \right)^{\frac{\beta}{p}} C' \leq \\
C_2 \left( \int_{\Omega} |Du_{h,k}^n - Du_h|^\beta dx \right)^{\frac{\beta}{p}} \left( \int_{\Omega} \left( 1 + |Du_{h,k}^n| + |Du_h| \right)^p dx \right)^{\frac{p-\beta}{p}} \leq \\
C_2 \int_{\Omega} \left( 1 + |Du_{h,k}^n| + |Du_h| \right)^{p-\beta} |Du_{h,k}^n - Du_h|^{\beta} dx \leq
\]
\[
\begin{align*}
\int_\Omega (c_h^k(x, Du_h^k(x)) - c_k^h(x, Du_h(x)), Du_h^k(x) - Du_h(x)) \, dx = \\
\int_\Omega (c_h^k(x, Du_h(x)) - c_k^h(x, Du_h(x)), Du_h^k(x) - Du_h(x)) \, dx \leq \\
\left( \int_\Omega \left| c_h^k(x, Du_h(x)) - c_k^h(x, Du_h(x)) \right|^q \, dx \right)^{\frac{1}{q}} \left( \int_\Omega \left| Du_h^k - Du_h \right|^p \, dx \right)^{\frac{1}{p}} \leq \\
\omega \left( \frac{1}{k} \right) \left( \int_\Omega \left| Du_h^k - Du_h \right|^p \, dx \right)^{\frac{1}{p}} \\
\omega \left( \frac{1}{k} \right) C'' \left( \int_\Omega \left| Du_h^k - Du_h \right|^p \, dx \right)^{\frac{1}{p}}.
\end{align*}
\]

Hence,
\[
\limsup_{k \to \infty} \left( \int_\Omega \left| Du_h^k - Du_h \right|^p \, dx \right)^{\frac{\beta - 1}{p}} \leq \omega \left( \frac{1}{k} \right) C'' \rightarrow 0 \quad (47)
\]
as \( k \to \infty \). By retracing the above inequalities we easily see that this is true even if \( p = \beta \). Thus, since \( \|D\|_{L^p(\Omega, \mathbb{R}^n)} \) is an equivalent norm on \( W_0^{1,p}(\Omega) \), we obtain (46).

**Term 2.** The fact that
\[
\lim_{k \to \infty} \lim_{k \to \infty} \left\langle g, u_h^k - u^k \right\rangle = \lim_{k \to \infty} \left\langle g, u_h^k - u^k \right\rangle = 0,
\]
is a direct consequence of (38).

**Term 3.** Let us prove that
\[
\lim_{k \to \infty} \|u^k - u\|_{W_0^{1,p}(\Omega)} = 0. \quad (48)
\]
Putting \( \phi = u^k - u \) into (41) and (33) we obtain that
\[
\begin{align*}
\int_\Omega (c^k(x, Du^k(x)) - c^k(x, Du(x)), Du^k(x) - Du(x)) \, dx = \\
\int_\Omega (c^*(x, Du(x)) - c^*(x, Du(x)), Du^k(x) - Du(x)) \, dx.
\end{align*}
\]
Taking into account the monotonicity of \( c^k \), the boundedness of \( (Du^k)_{k=1}^\infty \) in \( L^p(\Omega, \mathbb{R}^n) \), and using that, according to (37),
\[
c^*(x, Du(x)) - c^*(x, Du(x)) = c^*(x, Du(x)) - c^*(y^k_i, Du(x)) \quad \text{for all} \quad x \in \Omega_i,
\]
together with the fact that \( c^* \) satisfies (15), we find (similarly as we obtained (47)) that
\[
\left( \int_\Omega \left| Du^k - Du \right|^p \, dx \right)^{\frac{\beta - 1}{p}} \rightarrow 0 \quad (50)
\]
as \( k \to \infty \). Thus, (48) follows by using \( \|D\|_{L^p(\Omega, \mathbb{R}^n)} \) as (an equivalent) norm on \( W_0^{1,p}(\Omega) \).

**Step 3.** Next we prove that \( c_h^*(x, Du_h(x)) \rightarrow c^*(x, Du(x)) \) weakly in \( L^q(\Omega, \mathbb{R}^n) \).

If \( g \in (L^q(\Omega, \mathbb{R}^n))^* \), then
\[
\begin{align*}
\lim_{h \to \infty} \left| \left\langle g, c_h^*(x, Du_h(x)) \right\rangle - \left\langle g, c^*(x, Du(x)) \right\rangle \right| = \\
\lim_{k \to \infty} \lim_{h \to \infty} \left| \left\langle g, c_h^*(x, Du_h(x)) - c^*(x, Du(x)) \right\rangle \right| \leq
\end{align*}
\]
\[ \|g\| \lim_{k \to \infty} \limsup_{h \to \infty} \left\| c_h^k(x, Du_h(x)) - c_h^{k,h}(x, Du_h^{k,h}(x)) \right\|_{L^q(\Omega, \mathbb{R}^n)} + \]
\[ \lim_{k \to \infty} \limsup_{h \to \infty} \left\| \left( g, c_h^{k,h}(x, Du_h^{k,h}(x)) \right) - c^k(x, Du(x)) \right\| + \]
\[ \|g\| \lim_{k \to \infty} \left\| c^k(x, Du^k(x)) - c^*(x, Du(x)) \right\|_{L^q(\Omega, \mathbb{R}^n)}. \]

It is sufficient to prove that all three terms on the right hand side are zero.

**Term 1.** Let us show that
\[ \lim_{k \to \infty} \limsup_{h \to \infty} \left\| c_h^k(x, Du_h(x)) - c_h^{k,h}(x, Du_h^{k,h}(x)) \right\|_{L^q(\Omega, \mathbb{R}^n)} = 0. \] (51)

Clearly
\[ |a + b|^q \leq 2^q (|a|^q + |b|^q) \] (52)
for any \( a, b \in \mathbb{R}^n \). Hence,
\[
\int_{\Omega} \left| c_h^k(x, Du_h(x)) - c_h^{k,h}(x, Du_h^{k,h}(x)) \right|^q \, dx \leq \]
\[ 2^q \int_{\Omega} \left| c_h^{k,h}(x, Du_h^{k,h}(x)) - c_h^{k,h}(x, Du_h(x)) \right|^q \, dx + \]
\[ 2^q \int_{\Omega} \left| c_h^{k,h}(x, Du_h(x)) - c_h^k(x, Du_h(x)) \right|^q \, dx. \]

Thus, by applying the continuity conditions (12) and Hölder’s inequality to the first term on the right-hand side of this inequality and (14) to the second term, we obtain that
\[
\int_{\Omega} \left| c_h^k(x, Du_h(x)) - c_h^{k,h}(x, Du_h^{k,h}(x)) \right|^q \, dx \leq \]
\[ 2^q C_1 \left( \int_{\Omega} \left( 1 + |Du_h^{k,h}| + |Du_h| \right)^p \, dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |Du_h^{k,h} - Du_h|^p \, dx \right)^{\frac{1}{p}} + \]
\[ 2^q \omega \left( \frac{1}{K} \right) \int_{\Omega} 1 + |Du_h|^p \, dx. \]

Therefore, by (47), the boundedness of \((Du_h^{k,h})\) and \((Du_h)\) in \(L^p(\Omega, \mathbb{R}^n)\) and the fact that \(\omega(1/k) \to 0\), it follows that
\[ \limsup_{h \to \infty} \left\| c_h^k(x, Du_h(x)) - c_h^{k,h}(x, Du_h^{k,h}(x)) \right\|_{L^q(\Omega, \mathbb{R}^n)} \to 0 \] (53)
as \( k \to \infty \).

**Term 2.** The fact that
\[ \lim_{h \to \infty} \left\| \left( g, c_h^{k,h}(x, Du_h^{k,h}(x)) \right) - c^k(x, Du(x)) \right\| = 0 \]
is a direct consequence of (38).

**Term 3.** Let us show that
\[ \lim_{k \to \infty} \left\| c^k(x, Du^k(x)) - c^*(x, Du(x)) \right\|_{L^q(\Omega, \mathbb{R}^n)} = 0. \]

Since \(c^k \in \mathfrak{B}(C_1, C_2, p, \gamma, \beta)\), it holds that
\[ |c^k(x, \xi_1) - c^k(x, \xi_2)| \leq C_1 (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^{\alpha} \] (54)
for all \( x, \xi_1, \xi_2 \in \mathbb{R}^n \). By (52) we obtain that
\[
\int_{\Omega} \left| c^k(x, Du^k(x)) - c^*(x, Du(x)) \right|^q \, dx \leq \]
2^2 \int_{\Omega} \left| c^k(x, Du^k(x)) - c^k(x, Du(x)) \right|^q \, dx + 2^2 \int_{\Omega} \left| c^k(x, Du(x)) - c^*(x, Du(x)) \right|^q \, dx.

Using (54) and Hölder’s inequality in the first term on the left-hand side of this inequality, (49) and (15) to the second term, we see that

\int_{\Omega} \left| c^k(x, Du^k(x)) - c^*(x, Du(x)) \right|^q \, dx \leq

2^q C_1 \left( \int_{\Omega} \left( 1 + |Du^k| + |Du|^p \right) \, dx \right)^{\frac{q-1}{p}} \left( \int_{\Omega} |Du^k - Du|^p \, dx \right)^{\frac{q}{p}} + 2^q \omega(1/k) \int_{\Omega} 1 + |Du|^p \, dx.

Hence, by (50) and the facts that \( \omega(1/k) \to 0 \) and \( u^k \) is bounded in \( W^{1,p}_0(\Omega) \), we obtain that

\int_{\Omega} \left| c^k(x, Du^k(x)) - c^*(x, Du(x)) \right|^q \, dx \to 0

as \( k \to \infty \). \hfill \Box

**Proof of Proposition 2.** Let \( f \in W^{-1,q}(\Omega) \) and consider the problem

\[
\begin{align*}
\int_{\Omega} (c_h^k(x, Du_h(x)), D\phi(x)) \, dx &= \langle f, \phi \rangle \quad \text{for every } \phi \in W^{1,p}_0(\Omega), \\
u_h &\in W^{1,p}_0(\Omega),
\end{align*}
\]

(55) along with the limit problem

\[
\begin{align*}
\int_{\Omega} (c^*(x, Du(x)), D\phi(x)) \, dx &= \langle f, \phi \rangle \quad \text{for every } \phi \in W^{1,p}_0(\Omega), \\
u &\in W^{1,p}_0(\Omega).
\end{align*}
\]

(56)

By the definition of G-convergence, we only have to prove that

\[ v_h \rightharpoonup v \text{ weakly in } W^{1,p}_0(\Omega), \]

\[ c_h^k(x, Du_h) \rightharpoonup c^*(x, Du) \text{ weakly in } L^q(\Omega, \mathbb{R}^n). \]

 Exactly as we obtained (43) and (44) in the proof of Theorem 3.1 we find that there exist subsequences, \((v_h), (c_h^k(.), Du_h(.))\) and functions \(v_\ast \in W^{1,p}_0(\Omega), \eta_\ast \in L^q(\Omega, \mathbb{R}^n)\) such that

\[ v_h \rightharpoonup v_\ast \text{ weakly in } W^{1,p}_0(\Omega), \]

(57)

\[ c_h^k(., Du_h(.)) \rightharpoonup \eta_\ast \text{ weakly in } L^q(\Omega, \mathbb{R}^n), \]

(58)

as \( h \to \infty \). Hence, according to (55),

\[ \int_{\Omega} \langle \eta_\ast, D\phi \rangle \, dx = \langle f, \phi \rangle \text{ for every } \phi \in C^\infty_0(\Omega). \]

Again, due to the uniqueness of (56) we only need to prove that

\[ \eta_\ast = c^*(., Du_\ast(.)) \text{ for a.e. in } \Omega. \]

(59)

The claimed conclusion is now obtained by using similar arguments as we did in the proof of Proposition 1 by letting \( Z' \) be an open bounded set which is strictly contained in \( \Omega_1 \) or \( \Omega_2 \), putting \( \xi = 0 \), replacing \( W^{1,p}_0(Z') \) with \( W^{1,p}_0(Z) \), replacing \( Z \) with \( \Omega \) between eq. (28) to (29) and replacing \( Z \) with \( \Omega_1 \) or \( \Omega_2 \) in the rest of the proof. \hfill \Box
Acknowledgement. We are grateful to the anonymous reviewer for helping us to improve the presentation of this paper.

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