On the Sapondzhyan–Babuška Paradox.*

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Abstract
We consider the Lamé form of the elasticity system and demonstrate the famous Paradox, which can be obtained by the limit passage in the polygonal plates with very small sides as the length of sides goes to zero.

The phenomenon is that boundary conditions in the original problem and the limit problem are different. We provide the asymptotic and numerical analysis for the approximation of displacements, energies and moments, demonstrating the Paradox.

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Introduction

The Paradox of Sapondzhyan–Babuška (see [1], [3], [2], [4]) was discovered when studying the asymptotic behavior of solutions to an elasticity system in a thin polygonal plate (inscribed in a plate with smooth boundary) as the length of the side of the polygon tends to zero and the number of sides goes to infinity. Limit behavior and asymptotics of solutions in 2-dimensional case and 3-dimensional asymptotics were studied in [5] and [6], respectively. The number of papers devoted to this paradox is now quite large and one can find the references in [7] and [8]. Almost all of this items deal with 2-dimensional Kirchhoff models (see [10]). In [4] the Kirchhoff and the Reissner models were compared (see also [11], [12], [13]). Since the Kirchhoff model does not distinguish conditions of Hard support \((u_3 = 0, u_\tau = 0, \sigma_{\nu\nu}(u) = 0)\) and Simple support \((u_3 = 0, \sigma_{\nu\tau}(u) = 0, \sigma_{\nu\nu}(u) = 0)\), it is often preferred to use the Reissner model. It must be noted that the paradox can be discovered only in the case of hard support and cannot be seen in the case of simple support. The authors in [4] did not estimate errors of 2-dimensional approximation of 3-dimensional displacements, strains and stresses. This aspect of the problem has been discussed in [6].

In this paper we give a survey of the most important aspects of the Sapondzhyan–Babuška Paradox. An effort has been made making the presentation as illustrating and self-contained as possible. We believe that our presentation makes the material more readable to a broader audience, including scientists within the mathematical elasticity community, as well as pure mathematicians. For example, we show how the convergence of solutions imply the convergence of the total strain energy, resultant moment and twist stiffness, and we discover that the limit values are nonstandard and correspond to the Paradox. We also present some numerical results and compare these results with that of the corresponding limit problem.

1 Setting of the Problem in 3-dimensional case

Assume that \(\Omega_h\) is a thin 3-dimensional isotropic plate, i.e.

\[
\Omega_h = \{x = (y, z) : y = (y_1, y_2) \in \omega; |z| < \frac{h}{2}\},
\]

where \(\omega \subset \mathbb{R}^2\) is a domain bounded by simple smooth closed convex contours \(\Gamma_0\) and \(\Gamma_D\), and the thickness \(h \in (0, 1]\) of the plate is a small parameter.
Consider boundary value problem in $\Omega_h$:

$$
\begin{align*}
Lu &\equiv -\mu \nabla \cdot \nabla u - (\lambda + \mu) \nabla \nabla \cdot u = 0 \text{ in } \Omega_h, \\
u & = 0 \quad \text{on } T_D = \Gamma_D \times (-\frac{h}{2}, \frac{h}{2}), \\
u_3 & = 0, \quad u_\tau = 0, \quad \sigma_{\nu\nu}(u) = 0 \quad \text{on } T_0 = \Gamma_0 \times (-\frac{h}{2}, \frac{h}{2}), \\
\sigma_{13}(u) & = 0, \quad \sigma_{23}(u) = 0, \quad \sigma_{33}(u) = \pm \frac{g}{2} \quad \text{on } \omega^\pm = \omega \times \{ \pm \frac{h}{2} \}. 
\end{align*}
$$

In addition, $(\tau, \nu)$ are natural coordinates in a neighborhood of the contour $\Gamma_0$, $\tau$ is the length of a curve along $\Gamma_0$, and $\nu$ is the distance from $\Gamma_0$ in the direction of internal normal, i.e. $\nu > 0$ in $\omega$; $\mu > 0$, $\lambda \geq 0$ are the Lamé coefficients; $\nabla = \left( \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z} \right)$ is the gradient; $g = (g_1, g_2, g_3)$ is a vector of boundary loads. The Dekart coordinates of the strains and stresses are:

$$
\varepsilon_{kl}(u) = \frac{1}{2} \left( \frac{\partial u_k}{\partial y_l} + \frac{\partial u_l}{\partial y_k} \right), \quad \varepsilon_{k3}(u) = \frac{1}{2} \left( \frac{\partial u_k}{\partial z} + \frac{\partial u_3}{\partial y_k} \right), \quad k, l = 1, 2, 3
$$

$$
\varepsilon_{33}(u) = \frac{\partial u_3}{\partial z},
$$

$$
\sigma_{ij}(u) = 2\mu \varepsilon_{ij}(u) + \delta_{ij} \lambda \left( \varepsilon_{11}(u) + \varepsilon_{22}(u) + \varepsilon_{33}(u) \right), \quad i, j = 1, 2, 3,
$$

where $\delta_{ij}$ is a Kronecker symbol. Finally $N = (N_1, N_2, 0)$ and $T = (N_2, -N_1, 0)$ are normal and tangential unit vectors to the lateral surface $T_0$ of the plate, and

$$
u_\nu = N \cdot u, \quad u_\tau = T \cdot u, \quad \sigma_{\nu\nu}(u) = N \cdot \sigma(u) N, \quad \sigma_{\nu\tau}(u) = N \cdot \sigma(u) T.
$$

The boundary conditions in (1) mean that to the surfaces $\omega^\pm = \omega \times \{ \pm h/2 \}$ we apply antisymmetric transversal load, i.e. we have pure bending. The part $T_D$ is clamped and on $T_0$ we have hard support.

Let us construct the approximation of a contour $\Gamma_0$. Denote by $N$ a big natural number and by $P^1, \ldots, P^{N-1}, P^N \equiv P^0$ the points on the contour $\Gamma_0$, which separate it on the $N$ curves of the length $\rho = N^{-1}$. Assume that

$$
\rho = ah,
$$

where $a$ is fixed, in other words, the thickness of a plate is $h = a^{-1} \rho = a^{-1}N^{-1}$.

Connecting points $P^k$ and $P^{k+1}$ by segments we obtain a polygonal curve $\Gamma_N$. Consider a plate

$$
\Omega_{N,h} = \{(y, z) : y \in \omega_N; |z| < \frac{h}{2}\},
$$
where \( \omega_N \) is the 2-dimensional domain bounded by \( \Gamma_N \) and \( \Gamma_D \). Let us study a problem in \( \Omega_{N,h} \)

\[
\begin{aligned}
&L u_N^N = 0 \text{ in } \Omega_{N,h}, \\
u_N^N = 0 \quad \text{on } \mathcal{T}_D, \\
u_3^N = 0, \quad u_s^N = 0, \quad \sigma_{nm}(u^N) = 0 \quad \text{on } \mathcal{T}_N = (\Gamma_N \setminus \mathcal{P}) \times (-\frac{h}{2}, \frac{h}{2}), \\
\sigma_{13}(u^N) = 0, \quad \sigma_{23}(u^N) = 0, \quad \sigma_{33}(u^N) = \pm \frac{g}{2} \quad \text{on } \omega_N^\pm = \omega_N \times \left\{ \pm \frac{h}{2} \right\}.
\end{aligned}
\]

(6)

Here, \( \mathcal{P} \equiv \{ P^1, \ldots, P^N \} \), \( n \) and \( s \) are normal and tangential coordinates in a neighborhood of polygonal curve \( \Gamma_N \), in addition \( u_s^N \) and \( \sigma_{nm} \) are defined in analogues way as (4) (we change \( N, T \) by \( N, S \), the respective normal and tangential unit vectors to \( T_N \)).

The question is to calculate the approximate limit as \( N \to +\infty \).

2 Setting of the Problem in 2-dimensional case

Now let us consider 2-dimensional approximation of 3-dimensional thin plate. We keep the notation of the previous section, i.e. \( \omega \subset \mathbb{R}^2 \) is a domain bounded by simple smooth closed convex contours \( \Gamma_0 \) and \( \Gamma_D \). Assume that the displacement vector satisfies in \( \omega \):

\[
Lu \equiv -\mu \nabla \cdot \nabla u - (\lambda + \mu) \nabla \nabla \cdot u = 0 \text{ in } \omega.
\]

(7)
In addition $\mu > 0$, $\lambda \geq 0$ are the Lamé coefficients and $\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)$ is the gradient. The Dekart coordinates of the strains and stresses are:

$$
\varepsilon_{kl}(u) = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right),
$$

$$
\sigma_{ij}(u) = 2\mu \varepsilon_{ij}(u) + \delta_{ij}\lambda \left( \varepsilon_{11}(u) + \varepsilon_{22}(u) \right), \quad i, j = 1, 2, \quad (8)
$$

where $\delta_{ij}$ is a Kroneker symbol.

Let us consider the following boundary conditions:

$$
\begin{align*}
    u(x) &= 0, \quad x \in \Gamma_D; \\
    u_\nu(x) &= 0, \quad \sigma_{\nu\tau}(u; x) = 0, \quad x \in \Gamma_0.
\end{align*}
$$

The first condition means that we have a clamped plate along $\Gamma_D$ and the second condition characterizes the ideal contact (without take-off and without friction) of the lateral surface of the plate with absolutely stiff profile $\Gamma_0$.

Let us study a problem in $\omega_N$

$$
\begin{align*}
    Lu^N &= f \text{ in } \omega_N, \\
    u^N &= 0 \text{ on } \Gamma_D, \\
    u^N_\nu &= 0, \quad \sigma_{\nu\tau}(u^N) = 0 \text{ on } \Gamma_N = \left( \Gamma_N \setminus \mathcal{P} \right). \quad (9)
\end{align*}
$$

Here, $n$ and $s$ are normal and tangential coordinates in a neighborhood of polygonal curve $\Gamma_N$ with corner-points $\mathcal{P} \equiv \{ P^1, \ldots, P^N \}$.

The limit behavior of the solution $u^N$ is the following: denote by $u^\infty$ the limit of $u^N$ as $N \to +\infty$. The function $u^\infty$ satisfies the problem

$$
\begin{align*}
    Lu^\infty &= f \text{ in } \omega, \\
    u^\infty &= 0 \text{ on } \Gamma_D, \\
    u^\infty_\nu &= 0, \quad \sigma_{\nu\tau}(u^\infty; x) = ku^\infty k(\tau) \text{ on } \Gamma_0. \quad (10)
\end{align*}
$$

where

$$
    k = \pm \frac{1}{2} \left[ \lambda(1 - \frac{1}{\varkappa})(1 \mp 1) + 2\mu + \mu \left( 1 - \frac{1}{\varkappa} \pm (1 + \frac{1}{\varkappa}) \right) \right], \quad (11)
$$

$$
    \varkappa = \frac{\lambda + 3\mu}{\lambda + \mu} \quad \text{and} \quad k \text{ is the curvature of the contour } \Gamma_0. \quad \text{The upper row in the definition of } k \text{ is used if } \Gamma_0 \text{ is external contour and lower row is used if } \Gamma_0 \text{ is internal contour. This gives that}
$$

$$
    k = 2\mu
$$

if $\Gamma_0$ is external contour, and

$$
    k = (\lambda + \mu)(\frac{1}{\varkappa} - 1) = (\lambda + \mu)(\frac{\lambda + \mu}{\lambda + 3\mu} - 1) = -\frac{2\mu(\lambda + \mu)}{\lambda + 3\mu} \quad (12)
$$

if $\Gamma_0$ is internal contour.
3 Other types of boundary conditions

Let us briefly discuss other types of boundary conditions on $\Gamma_0$.

The Dirichlet conditions (clamped edge): In this case the paradox phenomenon cannot be found, i.e. the boundary conditions in the original problem

$$u_n^N = 0, \quad u_s^N = 0 \quad \text{on } \hat{\Gamma}_N$$

lead to the Dirichlet conditions in the limit problem

$$u_\nu^\infty = 0, \quad u_\tau^\infty = 0 \quad \text{on } \Gamma_0.$$

The Neumann conditions (free edge): In this case also the paradox phenomenon cannot be found, i.e. the boundary conditions in the original problem

$$\sigma_{ns}(u^N) = 0, \quad \sigma_{nn}(u^N) = 0 \quad \text{on } \hat{\Gamma}_N$$

lead to the Neumann conditions in the limit problem

$$\sigma_{\nu\tau}(u^\infty) = 0, \quad \sigma_{\nu\nu}(u^\infty) = 0 \quad \text{on } \Gamma_0.$$

The last possible condition is

$$u_s^N = 0, \quad \sigma_{nn}(u^N) = 0 \quad \text{on } \hat{\Gamma}_N.$$

This condition leads to the paradox analogues to one discovered in Problem (9) and (10), i.e.

$$u_\tau^N = 0, \quad \sigma_{\nu\nu}(u^\infty; x) = \pm \frac{1}{2} u_\nu^\infty k(\tau) \left[ \lambda(1 - \frac{1}{\kappa})(1 \mp 1) + 2\mu + \mu \left( 1 - \frac{1}{\kappa} \pm (1 + \frac{1}{\kappa}) \right) \right] \quad \text{on } \Gamma_0.$$

4 The regular case

As an example we consider the case when $\Gamma_D$ is the boundary of a disc $\Omega_1$ centered at 0 with radius $R_1$ and $\Gamma_N$ is a regular polygon of $N$ corners centered at 0 where the inscribed circle has radius $R_0$. Let us study the following problem in $\omega_N$

$$Lw^N = 0 \text{ in } \omega_N,$$

$$w^N = \alpha(-x_2, x_1) \quad \text{on } \Gamma_D,$$

$$w_n^N = 0, \quad \sigma_{ns}(w^N) = 0 \quad \text{on } \hat{\Gamma}_N = (\Gamma_N \setminus \mathcal{P}),$$

(13)
for some constant $\alpha$. For small values of $\alpha$ the boundary condition on $\Gamma_D$ can be interpreted as a rotation of angle $\alpha$. Note that the solution of $u^N$ (13) can be written as the sum $w^N = v + u^N$, where $v$ is a fixed function in $W^{1,2}(\omega_N)$ such that
\begin{align}
\begin{cases}
v = (-x_2, x_1) & \text{on } \Gamma_D, \\
v = 0 & \text{in } \Omega_2 \setminus \omega_N,
\end{cases}
\end{align}
(14)
where $\Omega_2$ is the disc centered at 0 with radius $R_2 > R_0$ such that $\Omega_1 \setminus \Omega_2 \subset \omega_N$, and $u^N$ is the solution of (9) for $f = -Lv$. An example of a function $v$ satisfying these properties is given in (39) below.

### 4.1 Strain energy and resultant moment

Let $W_N$ denote the strain energy
\[W_N = \frac{1}{2} \int_{\omega_N} e(w^N)\sigma(w^N) \, dx,\]
and let $M_N$ be related to the resultant moment per unit length about the $z$ axis of the forces acting on the boundary $\Gamma_D$ given by
\[M_N = \int_{\Gamma_D} (-x_2 F_1(w^N) + x_1 F_2(w^N)) \, ds,
\]
where $F(w^N) = (F_1(w^N), F_2(w^N))$ is the stress vector $F(w^N) = \sigma(w^N)N$ acting on the boundary $\partial \omega_N$ with outward unit normal $N$. 
It holds that

\[ M_N = \frac{2W_N}{\alpha} \quad (15) \]

Indeed, \( Lw^N = 0 \) is equivalent with the equation \( \text{div} \sigma(w^N) = 0 \). Thus, inserting \( \varphi = w^N \) into the Greens formula

\[
\int_{\omega_N} \varphi \cdot \text{div} \sigma(w^N) \, dx + \int_{\omega_N} e(\varphi)\sigma(w^N) \, dx = \int_{\partial\omega_N} \varphi \cdot F(w^N) \, ds, \quad (16)
\]

we obtain that

\[
W_N = \frac{1}{2} \int_{\omega_N} e(w^N)\sigma(w^N) \, dx = \frac{1}{2} \int_{\partial\omega_N} w^N \cdot F(w^N) \, ds = \frac{1}{2} \alpha \int_{\Gamma_D} (-x_2F_1(w^N) + x_1F_2(w^N)) \, ds + \frac{1}{2} \int_{\Gamma_N} w^N \cdot F(w^N) \, ds = \frac{\alpha M_N}{2} + \frac{1}{2} \int_{\Gamma_N} w^N \cdot F(w^N) \, ds. \quad (17)
\]

Letting \( N = (N_1, N_2) \) and \( T = (N_2, -N_1) \) denote the normal and tangential unit vectors to \( \Gamma_N \) we obtain that

\[
w^N \cdot (\sigma(w^N)N) = w^N \cdot F(w^N),
\]

i.e.

\[
w^N \cdot \sigma_{nn}(w^N) + w^N \cdot \sigma_{ns}(w^N) = w^N \cdot F(w^N). \quad (18)
\]

Thus,

\[
\int_{\Gamma_N} w^N \cdot F(w^N) \, ds = \int_{\Gamma_N} (w^N \cdot \sigma_{nn}(w^N) + w^N \cdot \sigma_{ns}(w^N)) \, ds = \int_{\Gamma_N} (0 \cdot \sigma_{nn}(w^N) + w^N \cdot 0) \, ds = 0,
\]

and, accordingly, (15) follows by (17). A different proof which is derived from the weak formulation is given in Section 4.3.
Due to linearity, \( w^N = \alpha w_0^N \), where \( w_0^N \) is the solution of (13) for the case \( \alpha = 1 \). Thus

\[
M_N = \alpha J_N
\]

(19)

where

\[
J_N = \int_{\Gamma_D} \left( -x_2 F_1(w_0^N) + x_1 F_2(w_0^N) \right) \, ds,
\]

which shows that the ratio \( M_N/\alpha = J_N \) is independent of the angle \( \alpha \). The constant \( J_N \) measures the resistance against rotation of outer boundary \( \Gamma_D \), and we therefore call it the twist stiffness. According to (17), this parameter is related to the strain energy \( W_N \) by the formula

\[
J_N = \frac{2W_N}{\alpha^2}.
\]

(20)

4.2 The limit problem

Let \( w_r \) and \( w_\theta \) denote displacement components of \( w \) in radial and circumferential directions, respectively. Let \( e_r(w) \) and \( e_\theta(w) \) be the unit elongation in radial and circumferential directions, respectively, let \( e_{r\theta}(w) \) denote the corresponding shear strain and let \( \gamma_{r\theta}(w) \equiv 2e_{r\theta}(w) \). Similarly, we let \( \sigma_{r}(w) \) and \( \sigma_{\theta}(w) \) denote the normal stress components in radial and circumferential directions, respectively, and let \( \sigma_{r\theta}(w) \) denote the corresponding shear stress. According to (10), the limit \( w = \lim_{N \to +\infty} w^N \) satisfies the problem

\[
\begin{align*}
Lw &= 0 \quad \text{in} \quad \omega, \\
\frac{\partial w_r}{\partial r} &= 0, \quad w_\theta = R_1 \alpha \quad \text{on} \quad \Gamma_D, \\
\frac{\partial w_r}{\partial r} &= 0, \quad \sigma_{r\theta}(w) = -w_\theta k \frac{1}{\rho_0} \quad \text{on} \quad \Gamma_0.
\end{align*}
\]

(21)

Here, we have used the fact \( \sigma_{r\theta}(w) = -\sigma_{\theta r}(w) \), which follows since the outward normal vector \( \mathbf{N} \) to \( \partial \omega \) is pointing in opposite direction of the radial unit vector. The solution of (21) can be found explicitly. The Hooks Law (8) takes the form

\[
\begin{align*}
\sigma_r(w) &= 2\mu e_r(w) + \lambda (e_r(w) + e_\theta(w)), \\
\sigma_\theta(w) &= 2\mu e_\theta(w) + \lambda (e_r(w) + e_\theta(w)), \\
\sigma_{r\theta}(w) &= 2\mu e_{r\theta}(w) = \mu \gamma_{r\theta}(w).
\end{align*}
\]

(22)

(23)

(24)

In polar coordinates the equilibrium equations \( Lw = 0 \) can be written as

\[
\begin{align*}
\frac{\partial \sigma_r(w)}{\partial r} + \frac{\partial \sigma_{r\theta}(w)}{r \partial \theta} + \frac{\sigma_r(w) - \sigma_\theta(w)}{r} &= 0, \\
\frac{\partial \sigma_\theta(w)}{r \partial \theta} + \frac{\partial \sigma_{r\theta}(w)}{\partial r} + \frac{2\sigma_{r\theta}(w)}{r} &= 0.
\end{align*}
\]
Moreover, it holds that
\[
e_r(w) = \frac{\partial w_r}{\partial r},
\]
\[
e_\theta(w) = \frac{w_r}{r} + \frac{\partial w_\theta}{\partial \theta},
\]
\[
\gamma_{r\theta}(w) = \frac{\partial w_r}{r} \frac{\partial w_\theta}{\partial \theta} + \frac{\partial w_\theta}{\partial r} - \frac{w_\theta}{r}
\]
(see e.g. [14, p. 66]). Motivated by the boundary conditions, we assume that \( w_r = 0 \) and that \( w_\theta \) is only dependent of \( r \) (below we will show that by this choice we will be able to find a displacement which satisfies the equilibrium equations and the boundary conditions, thus our assumption turns out to be correct). Hence, \( e_r(w) = 0, e_\theta(w) = 0, \)
\[
\gamma_{r\theta}(w) = \frac{\partial w_\theta}{\partial r} - \frac{w_\theta}{r} = r \frac{d (w_\theta / r)}{dr},
\]
and the equilibrium equations reduce to the single equation
\[
\frac{d \sigma_{r\theta}(w)}{dr} + \frac{2 \sigma_{r\theta}(w)}{r} = 0.
\]
Noting that the left side of this equation is equal to \( (1/r^2) d (r^2 \sigma_{r\theta}(w)) / dr, \) we obtain that
\[
\sigma_{r\theta}(w) = \frac{K}{r^2}
\]
for some constant \( K \). Alternatively, we can prove this by using that the resultant moment (per unit length) \( M \) about the \( z \) axis, of the forces acting on the boundary of the circle of radius \( r \), is given by
\[
M = r (2 \pi r \sigma_{r\theta}(w)),
\]
which implicitly shows that \( K = M/2 \pi \). Thus,
\[
\frac{d}{dr} \left( \frac{w_\theta}{r} \right) = \frac{\gamma_{r\theta}(w)}{r} = \frac{\sigma_{r\theta}(w)}{\mu r} = \frac{1}{2 \pi r^3 \mu} M.
\]
By integrating and implementing the boundary condition \( w_\theta(R_1) = R_1 \alpha \), we obtain that
\[
\frac{w_\theta(r)}{r} = \frac{M}{4 \pi \mu} \left( \frac{1}{R_1^2} - \frac{1}{r^2} \right) + \alpha. \tag{25}
\]
Next, using that
\[
-k \frac{w_\theta(R_0)}{R_0} = \sigma_{r\theta}(w) = \frac{M}{2 \pi R_0^2}, \tag{26}
\]
on $\Gamma_0$ (from the boundary conditions on $\Gamma_0$), we obtain from (25) that
\[
-\frac{M}{k^2\pi R_0^2} = \frac{M}{4\pi\mu} \left( \frac{1}{R_1^2} - \frac{1}{R_0^2} \right) + \alpha,
\]
i.e.
\[
\frac{M}{4\pi\mu} = \alpha \left( -\frac{1}{R_1^2} + \frac{1}{R_0^2} - \frac{2\mu}{kR_0^2} \right)^{-1}.
\]
Thus, we obtain the following solution
\[
\begin{align*}
walp &= 0 \\
\frac{w_\theta}{r} &= \alpha \left( -\frac{1}{R_1^2} + \frac{1}{R_0^2} - \frac{2\mu}{kR_0^2} \right)^{-1} \left( \frac{1}{R_1^2} - \frac{1}{r^2} \right) + 1.
\end{align*}
\]
Moreover, (25) implies that the corresponding twist stiffness $J = M/\alpha$ is given by the formula
\[
J = 4\pi\mu \left( -\frac{1}{R_1^2} + \frac{1}{R_0^2} - \frac{2\mu}{kR_0^2} \right)^{-1}.
\]
For both the 3-D case and the case of plain strain the Lame constants $\mu$ and $\lambda$ are related to the Youngs modulus $E$ and the Poisson ratio $\nu$ as follows
\[
\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}
\]
(see e.g. [9, p. 65]). Hence, by (12),
\[
k = -\frac{2\mu(\lambda + \mu)}{\lambda + 3\mu} = \frac{1}{(1+\nu)(4\nu - 3)}.
\]
Thus, according to (28),
\[
J = \frac{2\pi E}{\nu + 1} \left( -\frac{1}{R_1^2} + \frac{4}{R_0^2} (1 - \nu) \right)^{-1},
\]
In the case of plain stress we do the same with the only difference that one has to replace $\lambda$ by $\lambda^*$ defined by
\[
\lambda^* = \frac{2\lambda\mu}{\lambda + 2\mu} = \frac{E\nu}{1 - \nu^2}
\]
in the definition of $Lu$ (see e.g. [9, p. 95]).
4.3 Convergence of moments and energies

The resultant moment (per unit length) $M_N(r)$ about the $z$ axis, of the forces acting on the boundary of the circle of radius $r$, is given by

$$
M_N(r) = r\left(\int_0^{2\pi} \sigma_{r\theta}(w^N, r, \theta) \, r \, d\theta\right).
$$

Since no body forces are present, it is physically evident that $M_N(r) = M_N$ is constant. This fact can be verified rigorously from the weak formulation of (13),

$$
\int_{\omega_N} e(\varphi)\sigma(w^N) \, dx = 0, \quad \text{for all } \varphi \in V,
$$

where

$$
V = \left\{ \varphi \in W^{1,2}(\omega_N) : w^N = 0 \text{ on } \Gamma_D \text{ and } w^N_n = 0 \text{ on } \hat{\Gamma}_N \right\},
$$

which follows by (16) and (18). In order to show that $M_N(r)$ is constant, we first consider two disjoint intervals $I(p_1, r_1)$ and $I(p_2, r_2)$ in $I = [R_2, R_1]$ ($R_i$ is defined in the beginning of Section 4) of lengths $2r_1$ and $2r_2$ and with centers at some fixed points $p_1$ and $p_2$, respectively. Moreover, let $g = g(r)$ be a continuous function defined in $I$ as follows: $g(R_2) = 0$ and

$$
g'(r) = \begin{cases} 
s_1 & r \in I(p_1, r_1), \\
s_2 & r \in I(p_2, r_2), \\
0 & \text{elsewhere},
\end{cases}
$$

where $s_1$ and $s_2$ are constants satisfying the condition

$$
r_1s_1 + r_2s_2 = 0
$$

(by this condition $g$ we obtain that $g(R_1) = 0$). It is possible to prove that if $f \in L^1(I)$, and the identity

$$
\int_I f(r)g'(r) \, dr = 0
$$

holds for all disjoint intervals $I(p_1, r_1)$ and $I(p_2, r_2)$ in $I$, then there exists a constant $k$ such that $f(r) = k$ for almost every $r \in I$. Now, let $\varphi \in V$ with support in $\omega_N \setminus \Omega_2$ such that $\varphi_r = 0$ and $\varphi_\theta = \varphi_\theta(r)$ is defined by $\varphi_\theta(r)/r = g(r)$. By inserting $\varphi$ into (30) we obtain

$$
0 = \int_{\omega_N} e(\varphi)\sigma(w^N) \, dx =
$$
\[
\int_{\omega_N} (\sigma_r(w^N)\varepsilon_r(\varphi) + \sigma_\theta(w^N)\varepsilon_\theta(\varphi) + \sigma_{r\theta}(w^N)\gamma_{r\theta}(w)) \, dx = \\
\int_{\omega_N} \sigma_{r\theta}(w^N)\gamma_{r\theta}(w) \, dx = \int_0^{2\pi} \int_{R_2}^{R_1} \sigma_{r\theta}(w^N) \left( \frac{r \, d(\varphi_\theta/r)}{dr} \right) r \, dr \, d\theta = \\
\int_{R_2}^{R_1} \left( r \int_0^{2\pi} \sigma_{r\theta}(w^N) \, rdr \right) g'(r) \, dr,
\]
which gives that (33) is satisfied for

\[
f(r) = r \int_0^{2\pi} \sigma_{r\theta}(w^N) \, rdr = M_N(r).
\]

Hence, \( M_N(r) = M_N \) is constant. Now, let \( \Omega'_1 \) be the disc centered at 0 with radius \( R'_1 \) such that \( R_1 > R'_1 > R_2 \). Due to the invariance of \( M_N(r) = M_N \), we may express \( M_N \) in the following form

\[
M_N = (R'_1 - R_2)^{-1} \int_{R_2}^{R'_1} \int_0^{2\pi} r \sigma_{r\theta}(w^N) \, rdrdr = (R'_1 - R_2)^{-1} \int_{\Omega'_1 \setminus \Omega_2} r \sigma_{r\theta}(w^N) \, dx.
\]

Let us show that for a fixed \( \alpha \) we have that \( w^N \to w \) locally in \( W^{1,2}(\omega_N) \) as \( N \to +\infty \).

We use the statement (Theorem 3.6) from [5]. Suppose, that \( \chi \in C^\infty(\mathbb{R}) \) is a cut-off function, \( \chi(t) = 0 \) as \( |t| > \frac{1}{3} \) and \( \chi(t) = 1 \) as \( |t| < \frac{1}{6} \). Assume that \( \rho > 0 \) is sufficiently small, i.e. \( \chi_{\rho}(x) = \chi \left( \frac{x}{\rho} \right) \) is equal to zero near the boundary of the neighborhood, where the coordinates \((s, n)\) are defined. Also suppose, that

\[
\chi_j(\varepsilon, x) = \chi \left( \frac{|x - P^j|}{\varepsilon^2} \right), \quad X(\varepsilon, x) = 1 - \sum_{j=1}^{N} \chi_j(\varepsilon, x).
\]

**Lemma 4.1.** For solutions to the problems (9) and (10), respectively, the following estimate

\[
\|u^N - U^N\|_{W^{1,2}(\omega_N)} \leq c_\delta \varepsilon^{1-2\delta}
\]

is valid, where \( \delta > 0 \) and \( U^N \) is the global asymptotic expansion defined by the formula

\[
U^N(x) = X(\varepsilon, x) \left\{ u^\infty(x) + \varepsilon \chi_\rho(x)w(s, \frac{s}{\varepsilon}, \frac{n}{\varepsilon}) \right\} + \\
+ \sum_{j=1}^{N} \chi_j(\varepsilon, x) \left[ v_s(P^j) + \varepsilon C(P^j) \right] r \varepsilon^{\frac{1+\delta}{\rho}} \Phi^\pm(\varphi_j).
\]
Here $w$ is the boundary layer function (a solution of an auxiliary problem), $\Phi^\pm(\varphi)$ are trigonometric bounded functions (see [5, Sec.2]).

Note that the cut-off function $\chi_\rho$ is used for localization of the boundary layer. Thus, choosing $\rho$ and $\varepsilon$, we obtain $U^N = u^\infty$ in any given compact subset $K \subset \omega_N$ and consequently

$$
\|u^N - u^\infty\|_{W^{1,2}(K)} \longrightarrow 0 \quad \text{as} \quad N \to +\infty. \tag{37}
$$

Finally, from (37) we get the required convergence.

In particular, the convergence (35) implies that

$$
\sigma_{r\theta}(w^N) = \mu \left( \frac{\partial w_r^N}{r \partial \theta} - \frac{w_\theta^N}{r} \right)
$$

converges to $\sigma_{r\theta}(w)$ locally in $L^2(\omega_N)$. Hence, the right side of (34) converges to

$$
(R'_1 - R'_2)^{-1} \int_{\Omega'_1 \setminus \Omega_2} r \sigma_{r\theta}(w) \, dx = M,
$$

which shows that we have the following convergence of moments:

$$
M_N \to M
$$

as $N \to \infty$, as we might have guessed in the first place. Thus we find that the twist stiffness

$$
J_N \to J
$$

as $N \to \infty$.

The convergence of energies is however different from what we might believe as a first guess. Indeed, by inserting $\varphi = w$ into the Greens formula

$$
\int_\omega \varphi \cdot \text{div} \sigma(w) \, dx + \int_\omega e(\varphi) \sigma(w) \, dx = \int_\partial \omega \varphi \cdot F(w) \, ds,
$$

we obtain that

$$
W = \frac{1}{2} \int_\omega e(w) \sigma(w) \, dx = \frac{\alpha M}{2} + \frac{1}{2} \int_{\Gamma_0} w \cdot F(w) \, ds.
$$

Exactly as above, we find that

$$
\int_{\Gamma_0} w \cdot F(w) \, ds = \int_{\Gamma_0} (w_\nu \sigma_{\nu\nu}(w) + w_\tau \sigma_{\nu\tau}(w)) \, ds =
$$

$$
\int_{\Gamma_0} (0 \cdot \sigma_{\nu\nu}(w) + w_\theta \cdot (-\sigma_{r\theta}(w))) \, ds = - \int_{\Gamma_0} w_\theta \cdot (\sigma_{r\theta}(w)) \, ds.
$$
By (26), we have that

\[- \int_{\Gamma_0} w_\theta \cdot (\sigma_{r\theta}(w)) \, ds = - \int_{\Gamma_0} -\frac{M}{k2\pi R_0} \cdot \frac{M}{2\pi R_0^2} \, ds = \frac{M^2}{k2\pi R_0^2}.\]

Hence,

\[W = \frac{\alpha M}{2} + \frac{1}{2} \int_{\Gamma_0} w \cdot F(w) \, ds = \frac{\alpha M}{2} + \frac{M^2}{k4\pi R_0^2},\]

and since \( M = \alpha J \), we obtain the relation

\[\frac{2W}{\alpha^2} = \left( J + \frac{J^2}{k2\pi R_0^2} \right). \tag{38}\]

Since \( J_N = 2W_N/\alpha^2 \) converges to \( J \), we also have that \( W_N \) converges, but according to (38) the limit is not equal to \( W \).

As we promised earlier, we end this subsection by proving that \( J_N = 2W_N/\alpha^2 \), directly from the weak formulation (30). Motivated by (25) we let the function \( v \) be given by

\[v_\theta(r) = \alpha \left( \left( \frac{1}{R_2^2} - \frac{1}{R_1^2} \right)^{-1} \left( \frac{1}{R_1^2} - \frac{1}{r^2} \right) + 1 \right), \quad R_2 \leq r \leq R_1, \tag{39}\]

\[v_\theta(r) = 0 \quad \text{otherwise.}\]

This function clearly satisfies (14) since \( v_\theta(R_1) = R_1 \alpha \) and \( v_\theta(r) = 0 \) for \( r \leq R_2 \). By (30)

\[W_N = \frac{1}{2} \int_{\omega_N} e(w^N)\sigma(w^N) \, dx = \frac{1}{2} \int_{\omega_N} e(w^N + v)\sigma(w^N) \, dx = \]

\[\frac{1}{2} \int_{\omega_N} e(u^N)\sigma(u^N) \, dx + \frac{1}{2} \int_{\omega_N} e(v)\sigma(w^N) \, dx = \frac{1}{2} \int_{\omega_N} e(v)\sigma(w^N) \, dx = \]

\[\frac{1}{2} \int_{\omega_N} e(v)\sigma(w^N) \, dx = \frac{1}{2} \int_{\omega_N} \gamma_{r\theta}(v)\sigma_{r\theta}(w^N) \, dx = \]

\[\frac{1}{2} \int_{\omega_N} r \frac{d(v_\theta/r)}{dr} \sigma_{r\theta}(w^N) \, dx = \alpha \left( \frac{1}{R_2^2} - \frac{1}{R_1^2} \right)^{-1} \int_{R_2}^{R_1} \int_{0}^{2\pi} \frac{1}{r^2} \sigma_{r\theta}(w^N) \, r \, d\theta \, dr = \]

\[\alpha \left( \frac{1}{R_2^2} - \frac{1}{R_1^2} \right)^{-1} \int_{R_2}^{R_1} \frac{1}{r^3} \left( r \int_{0}^{2\pi} \sigma_{r\theta}(w^N) \, r \, d\theta \right) \, dr = \]

\[\alpha \left( \frac{1}{R_2^2} - \frac{1}{R_1^2} \right)^{-1} M_N \int_{R_2}^{R_1} \frac{1}{r^3} \, dr = \frac{1}{2} \alpha M_N \]

i.e. \( 2W_N = \alpha M_N \), which gives that \( J_N = 2W_N/\alpha^2 \).
4.4 Numerical results

Let us solve the problem (13) numerically for the case when $R_1 = 1$, $\Gamma_N = \Gamma_6$ is a regular hexagon with side-length equal to $1/\sqrt{3}$, i.e. $\theta_0 = \pi/6$ and $R_0 = 1/2$, $\alpha = 1$, $\nu = 0$ and $E = 1$. The numerical results are obtained by using the FE-program ANSYS 11.0. Due to rotation symmetry, it is enough to solve the problem on the set $\omega_0^N$ illustrated in Figure 2 by using periodic boundary condition on the radial and circumferential displacement components on the side surfaces denoted $\Delta_N$. More precisely we solve the following problem:

\[
\begin{align*}
Lw^N &= 0 \text{ in } \omega^N, \\
w^N &= (x_2, x_1) \text{ on } \Gamma_D^N, \\
w_n^N &= 0, \quad \sigma_{ns}(w^N) = 0 \text{ on } \Gamma_N, \\
w_r^N(r, -\theta_0) &= w_r^N(r, \theta_0), \quad w_\theta^N(r, -\theta_0) = w_\theta^N(r, \theta_0) \text{ on } \Delta_N.
\end{align*}
\] (40)

We have made an element mesh as shown in Figure 3, with increasing number of elements close to the corners of the hexagon. The corresponding deformed structure and the Von Mieses stresses are shown in Figure 4 and Figure 5, respectively. By using Ansys we may directly compute the strain energy $W_6' = W_N/N = W_6/6$ within $\omega_N$. The numerical value of this energy is $W_6' = 0.0601931$. Hence, the corresponding twist stiffness $J_6$ is

\[J_6 = \frac{2W_6}{\ell^2} = 12W_6' = 0.81312.\]
Similarly, computing the strain energy $W_N' = W_N/N$ for $N = 7, 8, \ldots$ we may obtain as many numerical values of $J_N = 2W_N = 2NW_N'$ as we want. For example, if $N = 30$ we obtain the strain energy $W_{30}' = 0.0130126$ by using the element mesh shown in Figure 6. Hence, we obtain the numerical value

$$J_{30} = \frac{2W_{30}}{1^2} = 60W_6' = (0.0130126)60 = 0.78076.$$  

We have not performed any numerical calculations for $N > 30$, due to the fact that $\omega_N'$ then gets to slender to obtain reliable numerical values. For the limit case we find from (29) that the corresponding twist stiffness $J$ takes the value

$$J = \frac{2\pi}{0 + 1} \left(-\frac{1}{(1)^2} + \frac{4}{(\frac{1}{2})^2}(1 - 0)\right)^{-1} = 0.41888.$$  

This indicates that the convergence $J_N \to J$ might be slow. Another possible reason for the observed deviation between $J_{30}$ and $J$ might be that the exact solution of the problem is to irregular around the lower corners $\omega_N'$ to be calculated with sufficient accuracy; in other words, the exact value of $J_{30}$ may be closer to $J$ than that calculated above. There are reasons to believe that other numerical approaches based on boundary layer methods or integral equations could give more reliable results that the classical FEM-method used in this paper.
References


Figure 6: Element mesh for the case $N = 30$ with 1983 elements and 6208 nodes.


